On perfect cones and absolute Baire-one retracts

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August 6, 2015

We introduce perfect cones over topological spaces and study their connection with absolute B_1-retracts.

1 Introduction

A subset E of a topological space X is a retract of X if there exists a continuous mapping $r : X \to E$ such that $r(x) = x$ for all $x \in E$. Different modifications of this notion in which r is allowed to be discontinuous (in particular, almost continuous or a Darboux function) were considered in [3, 10, 11, 12]. The author introduced in [5] the notion of B_1-retract, i.e. a subspace E of X for which there exists a Baire-one mapping $r : X \to E$ with $r(x) = x$ on E. Moreover, the following two results were obtained in [5].

Theorem 1.1. *Let X be a normal space and E be an arcwise connected and locally arcwise connected metrizable F_σ- and G_δ-subspace of X. If*
(i) E is separable, or
(ii) X is collectionwise normal,
then E is a B_1-retract of X.

Theorem 1.2. *Let X be a completely metrizable space and let E be an arcwise connected and locally arcwise connected G_δ-subspace of X. Then E is a B_1-retract of X.*

Note that in the above mentioned results E is a locally arcwise connected space. Therefore, it is naturally to ask

Question 1.3. *Is any arcwise connected G_δ-subspace E of a completely metrizable space X a B_1-retract of this space?*

In this work we introduce the notions of the perfect cone over a topological space and an absolute B_1-retract (see definitions in Section 2). We show that the perfect cone over a σ-compact metrizable zero-dimensional space is an absolute B_1-retract. Moreover, we give the negative answer to Question 1.3.

2 Preliminaries

Throughout the paper, all topological spaces have no separation axioms if it is not specified.

A mapping $f : X \to Y$ is *a Baire-one mapping* if there exists a sequence of continuous mappings $f_n : X \to Y$ which converges to f pointwise on X.

A subset E of a topological space X is called

- *a B_1-retract of* X if there exists a sequence of continuous mappings $r_n : X \to E$ such that $r_n(x) \to r(x)$ for all $x \in X$ and $r(x) = x$ for all $x \in E$; the mapping $r : X \to E$ is called *a B_1-retraction* of X onto E;
- *a σ-retract of* X if $E = \bigcup\limits_{n=1}^{\infty} E_n$, where $(E_n)_{n=1}^{\infty}$ is an increasing sequence of retracts of X;
- *ambiguous* if it is simultaneously F_σ and G_δ in X.

A topological space X is

- *perfectly normal* if it is normal and every closed subset of X is G_δ;
- *an absolute B_1-retract (in the class $\mathcal{C}$ of topological spaces)* if $X \in \mathcal{C}$ and for any homeomorphism h, which maps X onto a G_δ-subset $h(X)$ of a space $Y \in \mathcal{C}$, the set $h(X)$ is a B_1-retract of Y; in this paper we will consider only the case $\mathcal{C}$ is the class of all perfectly normal spaces;
- *a space with the regular G_δ-diagonal* if there exists a sequence $(G_n)_{n=1}^{\infty}$ of open neighborhoods of the diagonal $\Delta = \{(x, x) : x \in X\}$ in X^2 such that $\Delta = \bigcap\limits_{n=1}^{\infty} G_n = \bigcap\limits_{n=1}^{\infty} \overline{G}_n$;
- *contractible* if there exists a continuous mapping $\gamma : X \times [0, 1] \to X$ and $x^* \in X$ such that $\gamma(x, 0) = x$ and $\gamma(x, 1) = x^*$ for all $x \in X$; the mapping γ we call *a contraction*.

A family $(A_s : s \in S)$ of subsets of X is said to be *a partition of* X if $X = \bigcup\limits_{s \in S} A_s$ and $A_s \cap A_t = \emptyset$ for all $s \neq t$.

If a space X is homeomorphic to a space Y then we denote this fact by $X \simeq Y$.

For a mapping $f : X \times Y \to Z$ an a point $(x, y) \in X \times Y$ let $f^x(y) = f_y(x) = f(x, y)$.

3 Perfect cones and their properties

The cone $\Delta(X)$ *over a topological space* X is the quotient space $(X \times [0, 1])/(X \times \{0\})$ with the quotient mapping $\lambda : X \times [0, 1] \to \Delta(X)$. By v we denote *the vertex of the cone*, i.e. $v = \lambda(X \times \{0\})$. We call the set $\lambda(X \times \{1\})$ *the base of the cone.*

Let $X_1 = (0, 1)$ and $X_2 = [0, 1]$. Then $\Delta(X_2)$ is homeomorphic to a triangle $T \subseteq [0, 1]^2$, while $\Delta(X_1)$ is not even metrizable, since there is no countable base of neighborhoods at the cone vertex. Consequently, the naturally embedding of $\Delta(X_1)$ into $\Delta(X_2)$ is not a homeomorphism. Therefore, on the cone $\Delta(X)$ over a space X naturally appears one more topology $\mathcal{T}_p$ which coincides with the quotient topology $\mathcal{T}$ on $X \times (0, 1]$ and the base of neighborhoods of the vertex v forms the system $\{\lambda(X \times [0, \varepsilon)) : \varepsilon > 0\}$. The cone $\Delta(X)$ equipped with the topology $\mathcal{T}_p$ is said to be *perfect* and is denoted by $\Delta_p(X)$.

For all $x \in X$ we write

$$vx = \lambda^x([0, 1]).$$

Obviously,

$$\Delta(X) = \bigcup_{x \in X} vx.$$

It is easily seen that $vx \cap vy = \{v\}$ for all distinct $x, y \in X$ and $vx \simeq [0, 1]$ for all $x \in X$.

For every $y \in \Delta(X) \setminus \{v\}$ we set

$$\alpha(y) = \mathrm{pr}_X(\lambda^{-1}(y)). \tag{3.1}$$

Obviously, $\alpha : \Delta(X) \setminus \{v\} \to X$ is continuous in both topologies $\mathcal{T}$ and $\mathcal{T}_p$.

Let

$$\beta(y) = \begin{cases} \mathrm{pr}_{[0,1]}(\lambda^{-1}(y)), & y \neq v, \\ 0, & y = v. \end{cases} \tag{3.2}$$

Then $\beta : \Delta_p(X) \to [0, 1]$ is a continuous function. Indeed, it is evident that β is continuous on $\Delta_p(X) \setminus \{v\}$. Since

$$\beta^{-1}([0, \varepsilon)) = \lambda(X \times [0, \varepsilon)) \tag{3.3}$$

for any $\varepsilon > 0$, the set $\beta^{-1}([0, \varepsilon))$ is a neighborhood of v. Consequently, β is continuous at v.

We observe that

$$\lambda(\alpha(y), \beta(y)) = y$$

for all $y \in \Delta(X) \setminus \{v\}$.

Remark 3.1.

1. The concept of the perfect cone over a separable metrizable space was also defined in [8, p. 55].
2. We observe that $x \mapsto \lambda(x, 1)$ is a homeomorphism of X onto $\lambda(X, \times\{1\}) \subseteq \Delta_p(X)$. Therefore, we can identify X with its image and consider X as a subspace of $\Delta_p(X)$.
3. In the light of the previous observation we may assume that the mapping α defined by formula (3.1) is a retraction.
4. The system $\{\beta^{-1}([0, \varepsilon)) : \varepsilon > 0\}$ is the base of neighborhoods of the vertex of the cone according to (3.3).

Proposition 3.2. (cf. [8, p. 55]) *The cone $\Delta(X)$ over a compact space X is perfect.*

Proof. Let W be an open neighborhood of v in $\Delta(X)$. Then for every $x \in X$ there exist a neighborhood U_x of x and $\delta_x > 0$ such that $\lambda(U_x \times [0, \delta_x)) \subseteq W$. Choose a finite subcover $(U_1, \dots, U_n)$ of $(U_x : x \in X)$ and put $\varepsilon = \min\{\delta_1, \dots, \delta_n\}$. Then $\lambda(X \times [0, \varepsilon)) \subseteq W$. Hence, $\Delta(X)$ is the perfect cone. $\square$

Theorem 3.3. *Let X be a topological space.*

1. *If X is Hausdorff, then $\Delta_p(X)$ is Hausdorff.*
2. *If X is regular, then $\Delta_p(X)$ is regular.*
3. *If X is a countable regular space, then $\Delta_p(X)$ is perfectly normal.*
4. *$\Delta_p(X)$ is contractible.*
5. *If X is locally (arcwise) connected, then: a) $\Delta(X)$ is locally (arcwise) connected; b) $\Delta_p(X)$ is locally (arcwise) connected.*
6. *If X is metrizable, then $\Delta_p(X)$ is metrizable.*

Proof. 1). Let $x, y \in \Delta_p(X)$ and $x \neq y$. Since $\Delta_p(X) \setminus \{v\}$ is homeomorphic to the Hausdorff space $X \times (0, 1]$, it is sufficient to consider the case $x = v$ or $y = v$. Assume that $x = v$ and $y \neq v$. Then $0 = \beta(x) < \beta(y) \leq 1$, where β is defined by formula (3.2). Set $O_x = \lambda(X \times [0, \beta(y)/2))$ and $O_y = \lambda(X \times (\beta(y)/2, 1])$. Then O_x and O_y are disjoint neighborhoods of x and y in $\Delta_p(X)$, respectively.

2). Fix $y \in Y$ and a closed set $F \subseteq \Delta_p(X)$ such that $y \notin F$. Since $X \times (0, 1]$ is regular, the case $y \neq v$ and $v \notin F$ is obvious.

Let $y = v$. Choose $\varepsilon > 0$ such that $F \cap \lambda(X \times [0, \varepsilon)) = \emptyset$. Then $U = \lambda(X \times [0, \varepsilon/2))$ and $V = \lambda(X \times (\varepsilon/2, 1])$ are disjoint open neighborhoods of v and F in $\Delta_p(X)$, respectively.

Now let $y \neq v$ and $v \in F$. We take $\varepsilon > 0$ such that $O_v = \beta^{-1}([0, \varepsilon))$ is an open neighborhood of v with $y \notin \overline{O_v}$. Moreover, since $\Delta_p(X) \setminus \{v\}$ is regular, there exists an open neighborhood O_y of y in $\Delta_p(X)$ with $\overline{O_y} \cap G \subseteq \Delta_p \setminus F$. Then $U = O_y \setminus \overline{O_v}$ is an open neighborhood of y in $\Delta_p(X)$ such that $\overline{U} \cap F = \emptyset$.

3). We observe that $\Delta_p(X)$ is regular by the previous proposition. Moreover, Y is hereditarily Lindelöf (and, consequently, normal) as the union of countably many homeomorphic copies of $[0, 1]$. Hence, $\Delta_p(X)$ is perfectly normal.

4). For all $y \in \Delta_p(X)$ and $t \in [0, 1]$ define

$$\gamma(y, t) = \begin{cases} \lambda(\alpha(y), t \cdot \beta(y)), & y \neq v, \\ v, & y = v, \end{cases} \tag{3.4}$$

where α and β are defined by (3.1) and (3.2), respectively. Then $\gamma(y, 0) = v$ and $\gamma(y, 1) = y$ for all $y \in Y$. Clearly, γ is continuous on $(Y \setminus \{v\}) \times [0, 1]$. Let $\varepsilon > 0$

and $W = \lambda(X \times [0, \varepsilon)) = \beta^{-1}([0, \varepsilon))$. Since $\beta(\gamma(y, t)) = t \cdot \beta(y)$, $\gamma(W \times [0, 1]) \subseteq W$. Hence, γ is continuous at each point of the set $\{v\} \times [0, 1] = \beta^{-1}(0)$.

5a). Notice that the space $\Delta(X) \setminus \{v\}$ is locally (arcwise) connected, since it is homeomorphic to the locally (arcwise) connected space $X \times (0, 1]$. Let us check that $\Delta(X)$ is locally (arcwise) connected at v. Fix an open neighborhood W of v in $\Delta(X)$. Then $V = \lambda^{-1}(W)$ is open in $X \times [0, 1]$ and $X \times \{0\} \subseteq V$. For every $x \in X$ we denote by G_x the (arcwise) component of V with $(x, 0) \in G_x$. Then G_x is open in $\Delta(X) \setminus \{v\}$ and, consequently, in $\Delta(X)$. Let $G = \bigcup\limits_{x \in X} G_x$. Then the set $\lambda(G)$ in open neighborhood of v such that $\lambda(G) \subseteq W$. It remains to observe that $\lambda(G)$ is (arcwise) connected, since $v \in \lambda(G_x)$ and $\lambda(G_x)$ is (arcwise) connected for all $x \in X$.

5b). Let $\varepsilon > 0$ and $W = \beta^{-1}([0, \varepsilon))$. Since each element of W can be joined by a segment with v, W is an arcwise connected neighborhood of v.

6). Let ϱ be a metric generating the topology of X with $\varrho \leq 1$. For all $x, y \in X$ and $s, t \in [0, 1]$ we set

$$d(\lambda(x, s), \lambda(y, t)) = |t - s| + \min\{s, t\}\varrho(x, y).$$

Then d is a correctly defined, symmetric, nonnegative and nondegenerate mapping of $\Delta_p(X) \times \Delta_p(X)$. Moreover, the triangle inequality is satisfied, i.e.

$$d(\lambda(x, s), \lambda(z, u)) \leq d(\lambda(x, s), \lambda(y, t)) + d(\lambda(y, t), \lambda(z, u))$$

for all $(x, s), (y, t), (z, u) \in X \times [0, 1]$. If $t \geq \min\{s, u\}$, the inequality is obvious. Let $t < \min\{s, u\}$. Without loss of generality, we may assume that $t < u \leq s$. Then the above inequality is equivalent to

$$s - u + u\varrho(x, z) \leq s - t + t\varrho(x, y) + u - t + t\varrho(y, z),$$

i.e.,

$$\varrho(x, z) \leq 2(1 - \frac{t}{u}) + \frac{t}{u}(\varrho(x, y) + \varrho(y, z)),$$

which does hold since $\varrho(x, z) \leq 2$ and $\varrho(x, z) \leq \varrho(x, y) + \varrho(y, z)$. Moreover, d generates the topology of the perfect cone. It is obvious that the d-neighborhoods of v are the correct ones and the metric d on $\lambda(X \times (0, 1])$ is equivalent to the summing metric inherited from $X \times (0, 1]$. □

A subset E of a topological vector space X is *bounded* if for any neighborhood of zero U there is such $\gamma > 0$ that $E \subseteq \delta U$ for all $|\delta| \geq \gamma$.

Proposition 3.4. *Let Z be a topological vector space and $X \subseteq Z$ be a bounded set. Then $\Delta_p(X)$ is embedded to $Z \times \mathbb{R}$.*

Proof. Consider the set $C = \{(xt, t) : x \in X, t \in [0, 1]\}$. Let $\varphi(x, t) = (xt, t)$ for all $(x, t) \in X \times [0, 1]$ and $v^* = (0, 0) \in X \times [0, 1]$. Then the restriction $\varphi|_{X \times (0,1]}$ is a homeomorphism onto $C \setminus \{v^*\}$. Moreover, the mapping $\beta : C \to [0, 1]$,

$\beta(z,t) = t$, is continuous. Therefore, $\beta^{-1}([0,\varepsilon))$ is an open neighborhood of v^* in C for any $\varepsilon > 0$. Now we show that the system $\{\beta^{-1}([0,\varepsilon)) : \varepsilon > 0\}$ is a base of v^*. Take an open neighborhood of zero in Z, $\delta > 0$ and let $W = U \times (-\delta, \delta)$. Choose $\varepsilon \in (0, \delta)$ such that $tX \subseteq U$ for all t with $|t| < \varepsilon$. Then for each $y = (z,t) \in \beta^{-1}([0,\varepsilon))$ we have $|t| < \delta$ and $z \in tx \subseteq U$. Consequently, $\beta^{-1}([0,\varepsilon)) \subseteq W$. Hence, C is homeomorphic to the perfect cone $\Delta_p(X)$. □

The following result easily follows from [8, Theorem 1.5.9].

Corollary 3.5. *The cone $\Delta_p(X)$ over a finite Hausdorff space X is an absolute retract.*

4 Weak B_1-retracts

A subset E of a topological space X is called *a weak B_1-retract* of X if there exists a sequence of continuous mappings $r_n : X \to E$ such that $\lim\limits_{n\to\infty} r_n(x) = x$ for all $x \in E$. Clearly, every B_1-retract is a weak B_1-retract. The converse proposition is not true (see Example 4.7).

A space X is called *an absolute weak B_1-retract* if for any space Y and for any homeomorphic embedding $h : X \to Y$ the set $h(X)$ is a weak B_1-retract of Y.

Let $E = \bigcup\limits_{n=1}^{\infty} E_n$ and let $(r_n)_{n=1}^{\infty}$ be a sequence of retractions $r_n : X \to E_n$. If the sequence $(E_n)_{n=1}^{\infty}$ is increasing then $\lim\limits_{n\to\infty} r_n(x) = x$ for every $x \in E$. Thus, have proved the following fact.

Proposition 4.1. *Every σ-retract of a topological space X is a weak B_1-retract of X.*

Proposition 4.2. *Let X be a countable Hausdorff space. Then $\Delta_p(X)$ is an absolute weak B_1-retract.*

Proof. Assume that $\Delta_p(X)$ is a subspace of a topological space Z. Let $X = \{x_n : n \in \mathbb{N}\}$ and $X_n = \{x_1, \dots, x_n\}$. Then $\Delta_p(X) = \bigcup\limits_{n=1}^{\infty} \Delta_p(X_n)$ and every $\Delta_p(X_n)$ is a retract of Z by Corollary 3.5. Then $\Delta_p(X)$ is a weak B_1-retract of Z by Proposition 4.1. □

It was proved in [5] that a B_1-retract of a connected space is connected. It turns out that this is still valid for weak B_1-retracts.

Theorem 4.3. *Let X be a connected space. Then any weak B_1-retract E of X is connected.*

Proof. Let $(r_n)_{n=1}^{\infty}$ be a sequence of continuous mappings $r_n : X \to E$ such that $\lim\limits_{n\to\infty} r_n(x) = x$ for all $x \in E$. Denote $H = \bigcup\limits_{n=1}^{\infty} r_n(X)$. We show that H is connected. Conversely, suppose that $H = H_1 \cup H_2$, where H_1 and H_2

are disjoint sets which are closed in H. Observe that each set $B_n = r_n(X)$ is connected. Then $B_n \subseteq H_1$ or $B_n \subseteq H_2$. Choose an arbitrary $x \in H_1$. Then there exists a number n_1 such that $r_n(x) \in H_1$ for all $n \geq n_1$. Hence, $B_n \subseteq H_1$ for all $n \geq n_1$. Similarly, there exists a number n_2 such that $B_n \subseteq H_2$ for all $n \geq n_2$. Therefore, $B_n \subseteq H_1 \cap H_2$ for all $n \geq \max\{n_1, n_2\}$, which is impossible.

It is easy to see that $H \subseteq E \subseteq \overline{H}$. Since H and $\overline{H}$ are connected, E is connected too. □

Lemma 4.4. *Let X be a normal space, Y be a contractible space, $(F_i)_{i=1}^n$ be a sequence of disjoint closed subsets of X and let $g_i : X \to Y$ be a continuous mapping for every $1 \leq i \leq n$. Then there exists a continuous mapping $g : X \to Y$ such that $g(x) = g_i(x)$ on F_i for every $1 \leq i \leq n$.*

Proof. Let $y^* \in Y$ and $\gamma : Y \times [0, 1] \to Y$ be a continuous mapping such that $\gamma(y, 0) = y$ and $\gamma(y, 1) = y^*$ for all $y \in Y$. For all $x, y \in Y$ and $t \in [0, 1]$ define

$$h(x, y, t) = \begin{cases} \gamma(x, 2t), & 0 \leq t \leq 1/2, \\ \gamma(y, -2t + 2), & 1/2 < t \leq 1. \end{cases}$$

Then the mapping $h : Y \times Y \times [0, 1] \to Y$ is continuous, $h(x, y, 0) = x$ and $h(x, y, 1) = y$.

Let $n = 2$. By Urysohn's Lemma there is a continuous function $\varphi : X \to [0, 1]$ such that $\varphi(x) = 0$ on F_1 and $\varphi(x) = 1$ on F_2. For all $x \in X$ let

$$g(x) = h(g_1(x), g_2(x), \varphi(x)).$$

Clearly, $g : X \to Y$ is continuous and $g(x) = g_1(x)$ if $x \in F_1$, and $g(x) = g_2(x)$ if $x \in F_2$.

Assume the assertion of the lemma is true for k sets, where $k = 1, \ldots, n-1$, and prove it for n sets. According to our assumption, there exists a continuous mapping $\tilde{g} : X \to Y$ such that $\tilde{g}|_{F_i} = g_i$ for every $i = 1, \ldots, n-1$. Since the sets $F = \bigcup\limits_{i=1}^{n-1} F_i$ and F_n are closed and disjoint, there exists a continuous mapping $g : X \to Y$ such that $g|_F = \tilde{g}$ and $g|_{F_n} = g_n$. Then $g|_{F_i} = g_i$ for every $1 \leq i \leq n$. □

Theorem 4.5. *Let E be a contractible ambiguous weak B_1-retract of a normal space X. Then E is a B_1-retract of X.*

Proof. Let $(r_n)_{n=1}^\infty$ be a sequence of continuous mappings $r_n : X \to E$ such that $\lim\limits_{n\to\infty} r_n(x) = x$ for all $x \in E$. Choose increasing sequences $(E_n)_{n=1}^\infty$ and $(F_n)_{n=1}^\infty$ of closed subsets of X such that $E = \bigcup\limits_{n=1}^\infty E_n$ and $X \setminus E = \bigcup\limits_{n=1}^\infty F_n$. Fix $x^* \in E$. Then for every $n \in \mathbb{N}$ by Lemma 4.4 there exists a continuous mapping $f_n : X \to E$ such that $f_n(x) = r_n(x)$ if $x \in E_n$, and $f_n(x) = x^*$ if $x \in F_n$. It is easy to verify that the sequence $(f_n)_{n=1}^\infty$ is pointwise convergent on X and $\lim\limits_{n\to\infty} f_n(X) \subseteq E$. Let $r(x) = \lim\limits_{n\to\infty} f_n(x)$ for all $x \in X$. Then $r(x) = \lim\limits_{n\to\infty} r_n(x) = x$ for all $x \in E$. □

Proposition 4.6. *The perfect cone $\Delta_p(X)$ over a countable regular space X is an absolute B_1-retract.*

Proof. We first note that $\Delta_p(X)$ is perfectly normal by Theorem 3.3 (3). Assume that $\Delta_p(X)$ is a G_δ-subset of a perfectly normal space Z. Then $\Delta_p(X)$ is a weak B_1-retract of Z by Proposition 4.2. Moreover, $\Delta_p(X)$ is a contractible F_σ-subspace of Z. Hence, Theorem 4.5 implies that $\Delta_p(X)$ is a B_1-retract of Z. □

Let us observe that any B_1-retract of a space with a regular G_δ-diagonal is a G_δ-subset of this space [5, Proposition 2.2]. But it is not valid for weak B_1-retracts as the following example shows.

Example 4.7. *Let $\mathbb{Q}$ be the set of all rational numbers and $X = \mathbb{Q} \cap [0, 1]$. Then $\Delta_p(X)$ is a weak B_1-retract of $\mathbb{R}^2$, but is not a B_1-retract of $\mathbb{R}^2$.*

Proof. Indeed, $\Delta_p(X)$ is a weak B_1-retract of $\mathbb{R}^2$ by Proposition 4.2. Since $\Delta_p(X)$ is not a G_δ-set in $\mathbb{R}^2$, $\Delta_p(X)$ is not a B_1-retract. □

Theorem 4.8. *Let X be a perfectly normal space, E be a contractible G_δ-subspace of X, $x^* \in E$ and let $(E_n : n \in \mathbb{N})$ be a cover of E such that*

1. *$E_n \cap E_m = \{x^*\}$ for all $n \neq m$;*
2. *E_n is a relatively ambiguous set in E for every n;*
3. *E_n is a (weak) B_1-retract of X for every n.*

Then E is a (weak) B_1-retract of X.

Proof. From [6, p. 359] it follows that for every n there exists an ambiguous set C_n in X such that $C_n \cap E = E_n \setminus \{x^*\}$. Moreover, there exists a sequence $(F_n)_{n=1}^\infty$ of closed subsets of X such that $X \setminus E = \bigcup\limits_{n=1}^{\infty} F_n$. Let $D_n = C_n \cup F_n$, $n \geq 1$. Now define $X_1 = D_1$ and $X_n = D_n \setminus (\bigcup\limits_{k<n} D_k)$ if $n \geq 2$. Then $(X_n : n \in \mathbb{N})$ is a partition of $X \setminus \{x^*\}$ by ambiguous sets X_n and $X_n \cap E = E_n \setminus \{x^*\}$ for every $n \geq 1$.

Suppose that E_n is a weak B_1-retract of X for every n. Choose a sequence $(r_{n,m})_{m=1}^\infty$ of continuous mappings $r_{n,m} : X \to E_n$ such that $\lim\limits_{m\to\infty} r_{n,m}(x) = x$ for all $x \in E_n$. Since X_n is F_σ in X, for every n there is an increasing sequence $(B_{n,m})_{m=1}^\infty$ of closed subsets $B_{n,m}$ of X such that $X_n = \bigcup\limits_{m=1}^{\infty} B_{n,m}$. Let $A_{n,m} = \emptyset$ if $n > m$, and $A_{n,m} = B_{n,m}$ if $n \leq m$. Then Lemma 4.4 implies that for every $m \in \mathbb{N}$ there is a continuous mapping $r_m : X \to E$ such that $r_m|_{A_{n,m}} = r_{n,m}$ and $r_m(x^*) = x^*$.

We will show that $\lim\limits_{m\to\infty} r_m(x) = x$ on E. Fix $x \in E$. If $x = x^*$ then $r_m(x) = x$ for all m. If $x \neq x^*$ then there is a unique n such that $x \in E_n$. Since $(A_{n,m})_{m=1}^\infty$ increases, there exists a number m_0 such that $x \in A_{n,m}$ for

all $m \geq m_0$. Hence, $\lim\limits_{m\to\infty} r_m(x) = \lim\limits_{m\to\infty} r_{n,m}(x) = x$. Therefore, E is a weak B_1-retract of X.

If E_n is a B_1-retract of X for every n, we apply similar arguments. $\square$

5 Cones over ambiguous sets

Theorem 5.1. *Let $\Delta_p(X)$ be the perfect cone over a metrizable locally arcwise connected space X, Z be a normal space and let $h : \Delta_p(X) \to Z$ be an embedding such that $h(\Delta_p(X))$ is an ambiguous set in Z. If*

a) X is separable, or

b) $\Delta_p(X)$ is collectionwise normal,

then $h(\Delta_p(X))$ is a B_1-retract of Z.

Proof. We notice that $h(\Delta_p(X))$ is metrizable, arcwise connected and locally arcwise connected according to Theorem 3.3. Then the set $h(\Delta_p(X))$ is a B_1-retract of Z by Theorem 1.1. $\square$

By $B_\varepsilon(x_0)$ we denote an open ball in a metric space X with center at $x_0 \in X$ and with radius ε.

Theorem 5.2. *Let $\Delta_p(X)$ be the perfect cone over a zero-dimensional metrizable separable space X, Z be a normal space and let $h : \Delta_p(X) \to Z$ be such a homeomorphic embedding that $h(\Delta_p(X))$ is a closed set in Z. Then $h(\Delta_p(X))$ is a weak B_1-retract of Z.*

Proof. Without loss of generality we may assume that $\Delta_p(X)$ is a closed subspace of a normal space Z. Consider a metric d on X which generates its topological structure and (X, d) is a completely bounded space. For every $n \in \mathbb{N}$ there exists a finite set $A_n \subseteq X$ such that the family $\mathcal{B}_n = (B_{\frac{1}{n}}(a) : a \in A_n)$ is a cover of X. Since X is strongly zero-dimensional [2, Theorem 6.2.7], for every n there exists a finite cover $\mathcal{U}_n = (U_{i,n} : i \in I_n)$ of X by disjoint clopen sets $U_{i,n}$ which refines $\mathcal{B}_n$. Take an arbitrary $x_{i,n} \in U_{i,n}$ for every $n \in \mathbb{N}$ and $i \in I_n$. For all $x \in X$ and $n \in \mathbb{N}$ define

$$f_n(x) = x_{i,n},$$

if $x \in U_{i,n}$ for some $i \in I_n$. Then every mapping $f_n : X \to X$ is continuous and $\lim\limits_{n\to\infty} f_n(x) = x$ for all $x \in X$.

Fix $n \in \mathbb{N}$. For all $y \in \Delta_p(X)$ we set

$$g_n(y) = \begin{cases} \lambda(f_n(\alpha(y)), \beta(y)), & \text{if } y \neq v, \\ v, & \text{if } y = v. \end{cases}$$

We prove that $g_n : \Delta_p(X) \to \Delta_p(X)$ is continuous at $y = v$. Indeed, let $(y_m)_{m=1}^\infty$ be a sequence of points $y_m \in Y$ such that $y_m \to v$. Assume that $y_m \neq v$ for all m. Show that $g_n(y_m) \to v$. Fix $\varepsilon > 0$. Since $\beta(y_m) \to 0$, there is a number m_0

such that $\beta(y_m) < \varepsilon$ for all $m \geq m_0$. Then $g_n(y_m) = \lambda(f_n(\alpha(y_m)), \beta(y_m)) \in \lambda(X \times [0, \varepsilon))$ for all $m \geq m_0$. Hence, g_n is continuous at v.

Note that $g_n(\Delta_p(X)) \subseteq K_n$, where $K_n = \bigcup\limits_{i \in I_n} vx_{i,n}$. Since K_n is a compact absolute retract by Corollary 3.5, K_n is an absolute extensor. Taking into account that $\Delta_p(X)$ is closed in Z, we have that there exists a continuous extension $r_n : Z \to K_n$ of g_n.

It remains to show that $\lim\limits_{n\to\infty} r_n(y) = y$ for all $y \in \Delta_p(X)$. Fix $y \in \Delta_p(X)$. If $y = v$ then $r_n(y) = g_n(y) = v$ for all $n \geq 1$. Let $y \neq v$. Since $\lim\limits_{n\to\infty} f_n(\alpha(y)) = \alpha(y)$ and λ is continuous,

$$\lim_{n\to\infty} r_n(y) = \lim_{n\to\infty} \lambda(f_n(\alpha(y)), \beta(y)) = \lambda(\alpha(y), \beta(y)) = y.$$

Hence, $\Delta_p(X)$ is a weak B_1-retract of Z. $\square$

Theorem 5.3. *The perfect cone $\Delta_p(X)$ over a σ-compact zero-dimensional metrizable space X is an absolute B_1-retract.*

Proof. Assume that $\Delta_p(X)$ is a G_δ-subspace of a perfectly normal space Z.

Since X is σ-compact, there exists an increasing sequence $(F_n)_{n=1}^\infty$ of compact subsets of Z such that $X = \bigcup\limits_{n=1}^{\infty} F_n$. Since for every $n \in \mathbb{N}$ the set $F_{n+1} \setminus F_n$ is open in the zero-dimensional metrizable separable space F_{n+1}, there exists a partition $(B_{n,m} : m \in \mathbb{N})$ of $F_{n+1} \setminus F_n$ by relatively clopen sets $B_{n,m}$ in F_{n+1}. Let $\mathbb{N}^2 = (n_k, m_k : k \in \mathbb{N})$, $H_0 = F_1$ and let $H_k = B_{n_k,m_k}$ for every $k \in \mathbb{N}$. Then the family $(H_k : k = 0, 1, \dots)$ is a partition of X by compact sets H_k.

Fix $k \in \mathbb{N}$. Let $E_k = \Delta_p(H_k)$ be the perfect cone over zero-dimensional metrizable separable space H_k. Then E_k is a closed subset of Z. Therefore, E_k is a weak B_1-retract of Z by Theorem 5.2.

Since $\Delta_p(X) = \bigcup\limits_{k=1}^{\infty} E_k$, Theorem 4.8 implies that $\Delta_p(X)$ is a weak B_1-retract of Z. It remains to apply Theorem 4.5. $\square$

Theorem 5.4. *The perfect cone $\Delta_p(X)$ over a σ-compact space $X \subseteq \mathbb{R}$ is an absolute B_1-retract.*

Proof. Suppose that $\Delta_p(X)$ is a G_δ-subspace of a perfectly normal space Z. Since $\Delta_p(X)$ is σ-compact, $\Delta_p(X)$ is F_σ in Z.

Let $G = \mathrm{int}_{\mathbb{R}} X$, $F = X \setminus G$, $A = \Delta_p(G)$ and $B = \Delta_p(F)$. Since G and F are σ-compact sets, A and B are σ-compact sets too. Hence, A and B are ambiguous subsets of $\Delta_p(X)$. Consequently, A and B are ambiguous in Z. Since G is metrizable locally arcwise connected separable space, A is a B_1-retract of Z by Theorem 5.1. Since F is zero-dimensional metrizable σ-compact space, B is a B_1-retract of Z according to Theorem 5.3. Theorem 4.8 implies that the set $\Delta_p(X) = A \cup B$ is a B_1-retract of Z. $\square$

Note that the condition of σ-compactness of X in Theorems 5.2 and 5.3 is essential (see Example 6.4).

6 The weak local connectedness point set of B_1-retracts

Let (Y,d) be a metric space. A sequence $(f_n)_{n=1}^\infty$ of mappings $f_n : X \to Y$ is *uniformly convergent to a mapping* f *at a point* x_0 *of* X if for any $\varepsilon > 0$ there exists a neighborhood U of x_0 and $N \in \mathbb{N}$ such that

$$d(f_n(x), f(x)) < \varepsilon$$

for all $x \in U$ and $n \geq N$. We observe that if every f_n is continuous at x_0 and the sequence $(f_n)_{n=1}^\infty$ converges uniformly to f at x_0, then f is continuous at x_0.

By $R((f_n)_{n=1}^\infty, f, X)$ we denote the set of all points of uniform convergence of the sequence $(f_n)_{n=1}^\infty$ to the mapping f.

The closure of a set A in a subspace E of a topological space X we denote by $\overline{A}^{\,E}$.

A space X is *weakly locally connected at* $x_0 \in X$ if every open neighborhood of x_0 contains a connected (not necessarily open) neighborhood of x_0. The set of all points of weak local connectedness of X we will denote by $WLC(X)$.

Theorem 6.1. *Let X be a locally connected space, (E,d) be a metric subspace of X and let $r : X \to E$ be a B_1-retraction which is a pointwise limit of a sequence of continuous mappings $r_n : X \to E$. Then*

$$R((r_n)_{n=1}^\infty, r, X) \cap E \subseteq WLC(E).$$

Proof. Fix $x_0 \in R((r_n)_{n=1}^\infty, r, X) \cap E$ and $\varepsilon > 0$. Set $W = B_\varepsilon(x_0)$. Choose a neighborhood U_1 of x_0 in X and a number n_0 such that

$$d(r_n(x), r(x)) < \frac{\varepsilon}{4}$$

for all $x \in U_1$ and $n \geq n_0$. Since r is continuous at x_0, there exists a neighborhood $U_2 \subseteq X$ of x_0 such that

$$d(r(x), r(x_0)) < \frac{\varepsilon}{4}$$

for all $x \in U_2$. The locally connectedness of X implies that there is a connected neighborhood U of x_0 such that $U \subseteq U_1 \cap U_2$. Since $\lim\limits_{n\to\infty} r_n(x_0) = x_0$, there exists a number n_1 such that $r_n(x_0) \in U \cap E$ for all $n \geq n_1$. Let $N = \max\{n_0, n_1\}$ and

$$F = \overline{\bigcup_{n \geq N} r_n(U)}^{\,E}.$$

We show that $F \subseteq W$. Let $x \in U$ and $n \geq N$. Then

$$d(r_n(x), x_0) = d(r_n(x), r(x_0)) \leq d(r_n(x), r(x)) + d(r(x), r(x_0)) < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.$$

Thus, $r_n(x) \in B_{\varepsilon/2}(x_0)$. Then $\bigcup\limits_{n \geq N} r_n(U) \subseteq B_{\varepsilon/2}(x_0)$. Hence,

$$F \subseteq \overline{B_{\varepsilon/2}(x_0)}^{\,E} \subseteq W.$$

Moreover, $r(U) \subseteq F$, provided $\lim\limits_{n\to\infty} r_{N+n}(x) = r(x)$ for every $x \in U$. Observe that $U \cap E = r(U \cap E) \subseteq r(U)$. Therefore,

$$x_0 \in U \cap E \subseteq F \subseteq W,$$

which implies that F is a closed neighborhood of x_0 in E.

It remains to prove that F is a connected set. To obtain a contradiction, assume that $F = F_1 \cup F_2$, where F_1 and F_2 are nonempty disjoint closed subsets of F. Clearly, $F \cap U \neq \emptyset$.

Consider the case $F_i \cap U \neq \emptyset$ for $i = 1, 2$. The continuity of r_n implies that $r_n(U)$ is a connected set for every $n \geq 1$. Since $r_n(U) \subseteq F$, $r_n(U) \subseteq F_1$ or $r_n(U) \subseteq F_2$ for every $n \geq N$. Choose $x_i \in F_i \cap U$ for $i = 1, 2$. Taking into account that $\lim\limits_{n\to\infty} r_n(x_i) = x_i$ for $i = 1, 2$, we choose a number $k \geq N$ such that $r_n(x_i) \in F_i$ for all $n \geq k$ and for $i = 1, 2$. Then $r_k(U) \subseteq F_1 \cap F_2$, which implies a contradiction.

Now let $F_1 \cap U \neq \emptyset$ and $F_2 \cap U = \emptyset$. Then $U \cap E \subseteq F_1$. Since $r_n(x_0) \in U \cap E$, $r_n(x_0) \in F_1$, consequently, $r_n(U) \subseteq F_1$ for all $n \geq N$. Then $F \subseteq \overline{F_1} = F_1$. Therefore, $F_2 = \emptyset$, a contradiction. One can similarly prove that the case when $F_1 \cap U = \emptyset$ and $F_2 \cap U \neq \emptyset$ is impossible.

Hence, the set F is connected and $x_0 \in WLC(E)$. □

Note that we cannot replace the set $R((r_n)_{n=1}^\infty, r, X)$ by the wider set $C(r)$ of all points of continuity of the mapping r in Theorem 6.1 as the following example shows.

Example 6.2. *There exists an arcwise connected closed subspace E of $\mathbb{R}^2$ and a B_1-retraction $r : \mathbb{R}^2 \to E$ such that $C(r) \cap E \not\subseteq WLC(E)$.*

Proof. Let $a_0 = (0; 0)$, $a_n = (\frac{1}{n}; 0)$ for $n \geq 1$ and $X = \{a_n : n = 0, 1, 2, \ldots\}$. Denote by va_n the segment which connects the points $v = (1; 0)$ and a_n for every $n = 0, 1, \ldots$. Define $E = \bigcup\limits_{n=0}^{\infty} va_n$. Then E is an arcwise connected compact subspace of $\mathbb{R}^2$ and $WLC(E) = (E \setminus va_0) \cup \{v\}$. For all $x \in \mathbb{R}^2$ write

$$r(x) = \begin{cases} x, & \text{if } x \in E, \\ a_0, & \text{if } x \notin E. \end{cases}$$

It is easy to see that $r : \mathbb{R}^2 \to E$ is continuous at the point $x = a_0$. We show that $r \in B_1(\mathbb{R}^2, E)$. Since $X \setminus E$ is F_σ, choose an increasing sequence of closed subsets $X_n \subseteq \mathbb{R}^2$ such that $\mathbb{R}^2 \setminus E = \bigcup\limits_{n=1}^{\infty} X_n$. Let $E_n = \bigcup\limits_{k=0}^{n} va_k$, $n \geq 1$. For every $n \in \mathbb{N}$ define $A_n = X_n \cup E_n$. Then for every n the set A_n is closed in

$\mathbb{R}^2$, $A_n \subseteq A_{n+1}$ and $\bigcup\limits_{n=1}^{\infty} A_n = \mathbb{R}^2$. Clearly, the restriction $r|_{A_n} : A_n \to E_n$ is continuous for every n. By the Tietze Extension Theorem there is a continuous extension $f_n : \mathbb{R}^2 \to \mathbb{R}^2$ of $r|_{A_n}$ for every n. Notice that for every n there exists a retraction $\alpha_n : \mathbb{R}^2 \to E_n$. Let $r_n = \alpha_n \circ f_n$. Then $r_n : \mathbb{R}^2 \to E_n$ is a continuous mapping such that $r_n|_{A_n} = r|_{A_n}$ for every n.

It remains to show that $\lim\limits_{n\to\infty} r_n(x) = r(x)$ for all $x \in \mathbb{R}^2$. Indeed, fix $x \in \mathbb{R}^2$. Then there is a number N such that $x \in A_n$ for all $n \geq N$. Then $r_n(x) = r(x)$ for all $n \geq N$. Hence, $r \in B_1(\mathbb{R}^2, E)$. □

Theorem 6.3. *Let X be a locally connected Baire space and E be a metrizable B_1-retract of X. Then the set $E \setminus WLC(E)$ is of the first category in X.*

If, moreover, X has a regular G_δ-diagonal and E is dense in X then $WLC(E)$ is a dense G_δ-subset of X.

Proof. Let d be a metric on the set E which generates its topological structure. Consider a B_1-retraction $r : X \to E$ and choose a sequence $(r_n)_{n=1}^{\infty}$ of continuous mappings $r_n : X \to E$ such that $\lim\limits_{n\to\infty} r_n(x) = r(x)$ for all $x \in E$. Denote $R = R((r_n)_{n=1}^{\infty}, r, X)$. Then $R \cap E \subseteq WLC(E)$ by Theorem 6.1. According to Osgood's theorem [9], $X \setminus R$ is an F_σ-set of the first category in X. Hence, $E \setminus WLC(E)$ is a set of the first category in X.

Now assume that X has a regular G_δ-diagonal and $\overline{E} = X$. It follows from [5, Proposition 2.2] that E is G_δ in X. Moreover, the set R is dense in X, since X is Baire. Then $R \cap E$ is dense in X. Hence, $WLC(E)$ is dense in X. Observe that $WLC(E)$ is a G_δ-subset of E by [7, p. 233]. Then $WLC(E)$ is G_δ in X. □

The following example gives the negative answer to Question 1.3.

Example 6.4. *There exists an arcwise connected G_δ-set $E \subseteq \mathbb{R}^2$ such that E is the perfect cone over zero-dimensional metrizable separable space $X \subseteq \mathbb{R}$ and E is not a B_1-retract of $\mathbb{R}^2$.*

Proof. Let $\mathbb{I}$ be the set of irrational numbers and $X = \mathbb{I} \cap [0, 1]$. Define

$$E = \{(xt, t) : x \in X, t \in [0, 1]\}.$$

Then $E \simeq \Delta_p(X)$. Moreover, E is an arcwise connected G_δ-subset of $\mathbb{R}^2$. Clearly, $\overline{E} = [0, 1]^2$ and $WLC(E) = \{v\}$. Therefore, Theorem 6.3 implies that E is not a B_1-retract of $[0, 1]^2$. Consequently, E is not a B_1-retract of $\mathbb{R}^2$. □

7 Acknowledgements

The author would like to thank the referee for his helpful and constructive comments that greatly contributed to improving the final version of the paper.

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keywords: cone over a space, B_1-retract, Baire-one mapping
AMS Classification: Primary 54C20, 54C15; Secondary 54C50